

RIGIDITY AND TOLERANCE FOR PERTURBED LATTICES

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ABSTRACT. A perturbed lattice is a point process $\Pi = \{x + Y_x : x \in \mathbb{Z}^d\}$ where the lattice points in \mathbb{Z}^d are perturbed by i.i.d. random variables $\{Y_x\}_{x \in \mathbb{Z}^d}$. A random point process Π is said to be rigid if $|\Pi \cap B_0(1)|$, the number of points in a ball, can be exactly determined given $\Pi \setminus B_0(1)$, the points outside the ball. The process Π is called deletion tolerant if removing one point of Π yields a process with distribution indistinguishable from that of Π . Suppose that $Y_x \sim N_d(0, \sigma^2 I)$ are Gaussian vectors with d independent components of variance σ^2 . Holroyd and Soo showed that in dimensions $d = 1, 2$ the resulting Gaussian perturbed lattice Π is rigid and deletion intolerant. We show that in dimension $d \geq 3$ there exists a critical parameter $\sigma_r(d)$ such that Π is rigid if $\sigma < \sigma_r$ and deletion tolerant (hence non-rigid) if $\sigma > \sigma_r$.

1. INTRODUCTION

Let $\Pi = \{x + Y_x : x \in \mathbb{Z}^d\}$ denote the lattice \mathbb{Z}^d perturbed by independent and identically distributed random variables $\{Y_x\}_{x \in \mathbb{Z}^d}$ taking values in \mathbb{R}^d . In this paper we address the questions of rigidity and deletion tolerance of such point processes. Rigidity holds if given the points of Π outside a ball, one can determine exactly the number of points of Π inside that ball.

Deletion tolerance concerns the effect of removing a single point. If one point, say Y_0 , is removed from Π , can this be detected? More formally, are the laws of Π and $\Pi_u = \{x + Y_x : x \in \mathbb{Z}^d \setminus \{u\}\}$ mutually singular for $u \in \mathbb{Z}^d$?

Definition 1.1. A Π -point is an \mathbb{R}^d valued random variable Z such that $Z \in \Pi$ a.s. A point process Π is **deletion tolerant** if for any Π -point Z , the point process $\Pi \setminus Z$ is absolutely continuous¹ with respect to Π . The point process Π is **deletion singular** if Π and $\Pi \setminus Z$ are mutually singular for any Π -point Z . We say that Π is **insertion tolerant** if for any Borel set $V \subset \mathbb{R}^d$ with Lebesgue measure $\mathcal{L}(V) \in (0, \infty)$, if U is independent of Π and uniform in V then $\Pi \cup U$ is absolutely continuous with respect to Π . If Π and $\Pi \cup U$ are mutually singular for all such V , then we say that Π is **insertion singular**.

¹When discussing absolute continuity or singularity of two random objects, we are referring to their laws.

For a point process Π and a ball $B \subset \mathbb{R}^d$ we define $\Pi_{\text{in}} = \Pi_{\text{in}}(B) := \Pi \cap B$ and $\Pi_{\text{out}}(B) = \Pi \cap B^c$. We say that Π is **rigid** if for all balls $B \subset \mathbb{R}^d$ there exists a measurable function $N = N_B$ on the collection of discrete point sets in \mathbb{R}^d such that $N_B(\Pi_{\text{out}}(B)) = |\Pi_{\text{in}}(B)|$ a.s..

Rigidity turns out to be closely related to deletion tolerance where we consider removing multiple points. We write $\Pi_S := \{x + Y_x : x \in \mathbb{Z}^d \setminus S\}$.

Proposition 1.2. *If the distribution of Y_x has a density which is everywhere positive, then the perturbed lattice $\Pi = \{x + Y_x : x \in \mathbb{Z}^d\}$ is rigid if and only if Π and Π_S are mutually singular for all finite sets $S \subset \mathbb{Z}^d$.*

It was shown in [10] that the perturbed lattice is deletion singular in dimension $d = 1$ when the perturbations Y_x have bounded first moment and in dimension $d = 2$ when the perturbations have bounded second moment. In contrast, we show that when $d \geq 3$, the question of deletion tolerance depends more delicately on the law of the perturbations; in particular, for Gaussian perturbations it exhibits a phase transition.

Theorem 1.3. *Let Π be the perturbed lattice in \mathbb{Z}^d with Gaussian $N_d(0, \sigma^2 I)$ perturbations. For $d \geq 3$ there exist critical variances $0 < \sigma_r(d) \leq \sigma_c(d)$ such that*

- *If $\sigma > \sigma_c$ then Π is deletion tolerant and is mutually absolutely continuous with respect to Π_0 .*
- *If $0 < \sigma < \sigma_c$ then Π is deletion singular.*
- *If $0 < \sigma < \sigma_r$ then Π is rigid.*
- *If $\sigma > \sigma_r$ then Π is non-rigid.*

We conjecture that in fact $\sigma_c = \sigma_r$ and that for all i.i.d. perturbations, the perturbed lattice is rigid if and only if the perturbed lattice is deletion singular. However, in Theorem 1.6 we show that for similar point processes these notions may differ.

Given the results of [10], it is natural to ask if heavy tailed random variables with infinite means may be deletion tolerant. In the case of α -stable perturbations we give a complete characterization.

Theorem 1.4. *Let Π be a one dimensional perturbed lattice with symmetric α -stable perturbations. If $\alpha < 1$ then the perturbed lattice Π is deletion and insertion tolerant and mutually absolutely continuous with Π_0 , while if $\alpha \geq 1$ then it is deletion singular and rigid.*

In Section 3 we give a more general categorization of which perturbations give rise to deletion tolerance and rigidity.

1.1. Absolute Continuity. Assuming that the distribution of the perturbations has a density which is everywhere positive, we establish equivalences between the different notions of deletion and insertion tolerance.

Proposition 1.5. *If the distribution of Y_x has a density which is everywhere positive then the following are equivalent*

- (1) *The perturbed lattice is deletion tolerant.*
- (2) *The perturbed lattice is insertion tolerant.*
- (3) *The perturbed lattice is not deletion singular.*
- (4) *The perturbed lattice is not insertion singular.*
- (5) *The measures Π and Π_0 are mutually absolutely continuous.*
- (6) *The measures Π and Π_0 are not mutually singular.*

We will also consider the case where k points are inserted or deleted. Generalizing the earlier definitions, a point process Π is **k -deletion tolerant** if for any distinct Π points $Z_1, \dots, Z_k \in \Pi$, $\Pi \setminus Z$ is absolutely continuous with respect to Π and **k -deletion singular** if they are always mutually singular. We say that Π is **k -insertion tolerant** if for any Borel set $V \subset \mathbb{R}^d$ with Lebesgue measure $\mathcal{L}(V) \in (0, \infty)$, if U_1, \dots, U_k are independent points uniform in V and independent of Π then $\Pi \cup \{U_1, \dots, U_k\}$ is absolutely continuous with respect to Π . If Π and $\Pi \cup \{U_1, \dots, U_k\}$ are mutually singular then we say Π is **k -insertion singular**.

Perhaps surprisingly, there exists a translation-invariant point process $\hat{\Pi}$ that is deletion singular but not rigid. In fact, such a process can be 2-deletion tolerant; that is, removing a single point from $\hat{\Pi}$ yields a singular measure, but removing any two points from $\hat{\Pi}$ yields a process which is absolutely continuous to the original!

We construct $\hat{\Pi}$ as the union of two correlated perturbed lattices. For $d \geq 3$ and $0 < \delta < \sigma$, let Y_x be i.i.d. $N_d(0, (\sigma^2 - \delta^2)I)$ variables. For $(x, i) \in \mathbb{Z}^d \times \{1, 2\}$ let $Y'_{x,i}$ be i.i.d. $N_d(0, \delta^2 I)$ variables. Setting $\hat{Y}_{x,i} = Y_x + Y'_{x,i}$, we define the point process $\hat{\Pi} = \{x + \hat{Y}_{x,i} : (x, i) \in \mathbb{Z}^d \times \{1, 2\}\}$. The next theorem is proved in Section 7.

Theorem 1.6. *There exist $0 < \delta < \sigma$ such that $\hat{\Pi}$ is deletion singular but 2-deletion tolerant and hence non-rigid.*

1.2. Exponential Intersection Tails property. We say that a measure η on oriented paths in the lattice has **Exponential Intersection Tails** with parameter $0 < \theta < 1$, denoted $EIT(\theta)$, if for some $C > 0$,

$$\eta \times \eta \left\{ (\gamma, \gamma') : |\gamma \cap \gamma'| \geq n \right\} \leq C\theta^n.$$

The uniform measure on oriented paths on \mathbb{Z}^d has Exponential Intersection Tails for some $\theta < 1$ when $d \geq 4$ but not when $d = 3$. Moreover, in [2] a measure on oriented paths in \mathbb{Z}^3 was constructed with Exponential Intersection Tails while it was shown that no such measure exists when $d \leq 2$.

2. PROOF OF THEOREM 1.3

In this section we establish Theorem 1.3 by first proving results about more general perturbations. Let e_1, \dots, e_d denote the standard basis vectors in \mathbb{R}^d .

Proposition 2.1. *For $d \geq 3$ there exists $\rho(d) > 1$ such that the following holds. Let Π be a d -dimensional perturbed lattice with i.i.d. perturbations $\{Y_x\}_{x \in \mathbb{Z}^d}$ with density $g(x)$ which is everywhere positive. If*

$$\max_i \int_{\mathbb{R}^d} \left(\frac{g(x + e_i)}{g(x)} \right)^2 g(x) dx < \rho(d), \quad (2.1)$$

then the perturbed lattice Π is deletion tolerant.

Proof. Let η denote a distribution over oriented paths with $EIT(\theta)$ for some $\theta = \theta(d) \in (0, 1)$. Choose $\rho(d)$ so that $\rho(d)\theta(d) < 1$.

By the Cauchy-Schwartz inequality, the hypothesis of the theorem implies that

$$\max_{i,j} \int_{\mathbb{R}^d} \left(\frac{g(x + e_i)}{g(x)} \right) \left(\frac{g(x + e_j)}{g(x)} \right) g(x) dx < \rho(d). \quad (2.2)$$

Let $\gamma = \{\gamma_0, \gamma_1, \dots\} \subset \mathbb{Z}^d$ be an oriented walk on \mathbb{Z}^d from the origin, (i.e. $\gamma_0 = 0$ and $\gamma_i - \gamma_{i-1}$ is a standard basis vector). We define $Y^\gamma = \{Y_x^\gamma\}_{x \in \mathbb{Z}^d}$ to be a field of independent random variables distributed as

$$Y_x^\gamma \stackrel{d}{=} \begin{cases} Y_x + \gamma_{i+1} - \gamma_i & x \in \gamma, x = \gamma_i, \\ Y_x & x \notin \gamma. \end{cases}$$

By construction the point $\gamma_i + Y_{\gamma_i}^\gamma$ has the same distribution as $\gamma_{i+1} + Y_{\gamma_{i+1}}$ so changing the perturbations in this way has the effect of shifting the points on γ one step along the path. As there is then a point centered at every vertex in \mathbb{Z}^d except 0, it follows that the point process $\{x + Y_x^\gamma : x \in \mathbb{Z}^d\}$ has the same law as Π_0 .

Denote by ν and ν^γ the distributions of \tilde{Y} and Y^γ , respectively. We would be done if Y and Y^γ were mutually absolutely continuous, but of course they are singular, since we have altered significantly the distribution of a specific infinite sequence of points. However, using a similar argument to that of [1] and [3], this singularity can be smoothed away by averaging

over γ . Let Γ denote a random path with the law η satisfying $EIT(\theta)$, and define $\tilde{Y} = Y^\Gamma$. Thus if $\tilde{\nu}$ denotes the distribution of \tilde{Y} , then

$$\tilde{\nu} = \int \nu^\gamma \eta(d\gamma).$$

We will now show that when σ is sufficiently large, the distributions of ν and $\tilde{\nu}$ are mutually absolutely continuous. Denote $Y(m) = \{Y_x\}_{|x| \leq m}$ and let ν denote the measure induced by Y and ν_m the measure induced by $Y(m)$. Define $\tilde{Y}(m)$, $\tilde{\nu}$, $\tilde{\nu}_m$, $Y^\gamma(m)$, ν^γ and ν_m^γ similarly. We let $L(y)$ denote the Radon-Nikodym derivative $\frac{d\tilde{\nu}}{d\nu}$ and let $L_m(y)$ denote $\frac{d\tilde{\nu}_m}{d\nu_m}$. Observe that $L_m(Y_m)$ is a martingale which converges to $L(Y)$ almost surely. If $L_m(Y_m)$ is an L^2 bounded martingale then $\tilde{\nu} \ll \nu$ (see, e.g., [12], Theorem 12.32 or [13].) By definition of L_m ,

$$\begin{aligned} \mathbb{E}[L_m(Y_m)]^2 &= \int [L_m(y)]^2 \nu_m(dy) \\ &= \int \left[\int \frac{d\nu_m^\gamma}{d\nu_m}(y) \eta_m(d\gamma) \right]^2 \nu_m(dy) \\ &= \int \int \int \frac{d\nu_m^\gamma}{d\nu_m}(y) \frac{d\nu_m^{\gamma'}}{d\nu_m}(y) \eta_m(d\gamma) \eta_m(d\gamma') \nu_m(dy) \\ &= \int \int \int \frac{d\nu_m^\gamma}{d\nu_m}(y) \frac{d\nu_m^{\gamma'}}{d\nu_m}(y) \nu_m(dy) \eta_m(d\gamma) \eta_m(d\gamma'). \end{aligned} \quad (2.3)$$

For fixed γ the measure ν^γ is a product measure, so

$$\int \frac{d\nu_m^\gamma}{d\nu_m}(y) \frac{d\nu_m^{\gamma'}}{d\nu_m}(y) \nu_m(dy) = \prod_{x \in \mathbb{Z}^d: |x| \leq m} \int_{\mathbb{R}^d} \frac{d\nu_{m,x}^\gamma}{d\nu_{m,x}}(y_x) \frac{d\nu_{m,x}^{\gamma'}}{d\nu_{m,x}}(y_x) \nu_{m,x}(dy_x), \quad (2.4)$$

where $\nu_{m,x} = \mu$ is the distribution of Y_x which is simply μ . If $x \notin \gamma$, then $\frac{d\nu_{m,x}^\gamma}{d\nu_{m,x}} = 1$ and hence

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{d\nu_{m,x}^\gamma}{d\nu_{m,x}}(y_x) \frac{d\nu_{m,x}^{\gamma'}}{d\nu_{m,x}}(y_x) \nu_{m,x}(dy_x) &= \int_{\mathbb{R}^d} \frac{d\nu_{m,x}^{\gamma'}}{d\nu_{m,x}}(y_x) \nu_{m,x}(dy_x) \\ &= \int_{\mathbb{R}^d} \nu_{m,x}^{\gamma'}(dy_x) = 1. \end{aligned}$$

A similar result holds when $x \notin \gamma'$, so it remains to consider $x \in \gamma \cap \gamma'$. In this case $x = \gamma_{|x|} = \gamma'_{|x|}$ and for some $1 \leq j, j' \leq d$, we have $e_j = \gamma_{|x|+1} - \gamma_{|x|}$ and $e_{j'} = \gamma'_{|x|+1} - \gamma'_{|x|}$. Then by definition of $\tilde{\nu}_{m,x}$ and equation (2.2) we have that

$$\int_{\mathbb{R}^d} \frac{d\nu_{m,x}^\gamma}{d\nu_{m,x}}(y_x) \frac{d\nu_{m,x}^{\gamma'}}{d\nu_{m,x}}(y_x) \nu_{m,x}(dy_x) = \int_{\mathbb{R}^d} \left(\frac{g(x+e_j)}{g(x)} \right) \left(\frac{g(x+e_{j'})}{g(x)} \right) g(x) dx \leq \rho.$$

Defining $N = N_{\gamma, \gamma'} = |\gamma \cap \gamma'|$ and substituting in equation (2.4) we have that

$$\int \frac{d\nu_m^\gamma(y)}{d\nu_m}(y) \frac{d\nu_m^{\gamma'}(y)}{d\nu_m}(y) \nu_m(dy) \leq \rho^N$$

and so by equation (2.3)

$$\sup_m \mathbb{E}[L_m(Y_m)]^2 \leq \mathbb{E}\rho^N < \infty,$$

by the $EIT(\theta)$ assumption on η . It follows that $L_m(Y_m)$ converges to $L(Y)$ almost surely which is finite ν -almost everywhere and hence that $\tilde{\nu}$ is absolutely continuous with respect to ν . Since $\tilde{\nu}$ generates the point process Π_0 it follows that Π_0 is absolutely continuous with respect to Π . The result then follows by Proposition 1.5. \blacksquare

2.1. Deletion intolerance for small perturbations. In this section we show that if the perturbations are small enough then we have deletion intolerance. Let $\gamma = (u_0, u_1, \dots)$ denote a nearest neighbor path in \mathbb{Z}^d with $u_0 = 0$ and let $\gamma_n = (u_0, \dots, u_n)$. For an i.i.d. field $\{Y_u\}_{u \in \mathbb{Z}^d}$ let $M_{n,d}$ denote

$$M_{n,d} := \sup_{\gamma} \sum_{u \in \gamma_n} Y_u.$$

and

$$M_d := \limsup \frac{1}{n} M_{n,d}.$$

Since M_d is not affected by changing a finite number of Y_u it is almost surely constant depending only on the distribution of Y_u so we will denote this constant as $M_d(Y)$. A simple union bound over paths implies that M_d is finite when Y is Gaussian while Theorem 1 of [6] implies that $M_d(Y)$ is finite provided that

$$\mathbb{E} Y^d \log^{d+\varepsilon} Y < \infty. \quad (2.5)$$

We have the following result when $M_d(|Y|_1) < \frac{1}{2}$.

Lemma 2.2. *Suppose that Y_x has an absolutely continuous distribution with respect to Lebesgue measure on \mathbb{R}^d , the ℓ^1 norm $|Y_x|_1$ satisfies equation (2.5) and $M_d(|Y_x|_1) < \frac{1}{2}$. Then the perturbed lattice with perturbations $\{Y_x\}_{x \in \mathbb{Z}^d}$ is k -deletion singular for all $k \geq 1$.*

Proof. We consider only the case of $k = 1$, the case of larger k following essentially without change. With $\gamma = (u_0, u_1, \dots)$ and γ_n defined as above, for a countable set of points $A \subset \mathbb{Z}^d$ define

$$f(A) = \inf_{\psi: \mathbb{Z}^d \rightarrow A} \sup_{\gamma} \limsup_n \frac{1}{n} \sum_{u \in \gamma_n} |\psi(u) - u|_1,$$

where the supremum is over all paths γ and the infimum is over all bijections from \mathbb{Z}^d into A . Taking $A = \Pi$ and $\psi(u) = u + Y_u$ we have that

$$f(A) \leq \sup_{\gamma} \limsup_n \frac{1}{n} \sum_{u \in \gamma_n} |Y_u|_1 < \frac{1}{2},$$

since $M_d(|Y_u|_1) < \frac{1}{2}$.

Now consider $f(\Pi_0)$. We define the bijection $W : \Pi \rightarrow \mathbb{Z}^d$ so that $y = W(y) + Y_{W(y)}$; this is uniquely defined almost surely since the Y_x have distributions with no atoms. Given a bijection $\psi : \mathbb{Z}^d \rightarrow \Pi_0$, construct a path γ as follows. Let $v_0 = 0$ and set $v_{j+1} = W(\psi(v_j))$ for $j \geq 1$ and let $s_j = \sum_{k=1}^j |v_k - v_{k-1}|_1$. Suppose that $v_j = v_{j'}$ for some $j' > j$. Then

$$\psi(v_{j-1}) = W^{-1}(v_j) = W^{-1}(v_{j'}) = \psi(v_{j'-1})$$

and so $v_{j-1} = v_{j'-1}$. Iterating we have that $0 = v_0 = v_{j'-j}$ which is a contradiction since $v_{j'-j} \in W(\Pi_0) = \mathbb{Z}^d \setminus \{0\}$.

Let γ be a nearest neighbor path constructed by sequentially joining the v_i with the shortest intermediate paths, that is $\gamma = (u_0, \dots)$ satisfies $u_0 = v_0 = 0$ and $u_{s_j} = v_j$. Then since $Y_{v_{k+1}} = W^{-1}(v_{k+1}) - v_{k+1} = \psi(v_k) - v_{k+1}$ we have that,

$$\begin{aligned} \limsup_n \frac{1}{n} \sum_{u \in \gamma_n} |\psi(u) - u|_1 &\geq \limsup_j \frac{1}{s_j} \sum_{k=0}^j |\psi(v_k) - v_k|_1, \\ &\geq \limsup_j \frac{1}{s_j} \sum_{k=0}^j |v_{k+1} - v_j|_1 - |v_{k+1} - \psi(v_k)|_1 \\ &\geq \limsup_j \frac{1}{s_j} \left(s_j - \sum_{k=0}^j |Y_{v_{k+1}}|_1 \right) \\ &\geq 1 - \limsup_n \frac{1}{n} \sum_{u \in \gamma_n} |Y_u|_1 > \frac{1}{2}. \end{aligned}$$

Since almost surely $f(\Pi) < \frac{1}{2}$ and $f(\Pi_0) > \frac{1}{2}$ we have that the two measures are mutually singular. \blacksquare

2.2. Proof of Theorem 1.3.

Proof. If $g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)}$ is a one-dimensional Gaussian $N(0, \sigma^2)$ density then

$$\begin{aligned} \int_{\mathbb{R}} \left(\frac{g(x+1)}{g(x)} \right)^2 g(x) dx &= \int_{\mathbb{R}} \left(\frac{\exp[-(x-1)^2/2\sigma^2]}{\exp[-x^2/2\sigma^2]} \right)^2 g(x) dx \\ &= \int_{\mathbb{R}} \exp[(2x-1)/\sigma^2] g(x) dx \\ &= \exp[1/\sigma^2]. \end{aligned} \quad (2.6)$$

As the d -dimensional Gaussian measure with density $g_d(x)$ is a product measure, when calculating (2.1) the contributions to the product not in the direction of e_i cancel and the equation reduces to

$$\int_{\mathbb{R}^d} \left(\frac{g_d(x+e_i)}{g_d(x)} \right)^2 g_d(x) dx = \int_{\mathbb{R}} \left(\frac{g(x+1)}{g(x)} \right)^2 g(x) dx = \exp[1/\sigma^2].$$

It follows from Proposition 2.1 that for sufficiently large σ , the process Π is deletion and insertion tolerant and mutually absolutely continuous with Π_0 .

We now consider the case when σ is small. By scaling, the greedy lattice animal with weights $|Y_x|_1$ has a finite limiting value with $M(|Y_x|_1)$ proportional to σ . It follows by Lemma 2.2 that Π is deletion singular for sufficiently small $\sigma > 0$.

The existence of a critical value $\sigma_c(d)$ follows from the observation that increasing σ is equivalent to a semigroup acting on Π by shifting the points according to independent Brownian motions. If Π and Π_0 are not singular for some value of σ then they can be coupled with positive probability and hence they can be coupled for all larger values of σ as well. Hence, by Proposition 1.5, there must be a critical $\sigma_c(d)$ with deletion tolerance for $\sigma > \sigma_c(d)$ and deletion singularity for $\sigma < \sigma_c(d)$.

We similarly have that for each k there exists a threshold $\sigma_c(k, d)$ with k -deletion tolerance above $\sigma_c(k, d)$ and k -deletion singularity below. Letting $\sigma_r(d) = \inf_k \sigma_c(k, d)$ by Proposition 6.1 when $\sigma > \sigma_r$ there is some k for which Π is not k -deletion singular and hence not rigid by Proposition 1.2. Conversely, if $\sigma < \sigma_r$, then Π is k -deletion singular for all k and hence rigid by Propositions 6.1 and 1.2. It follows from Lemmas 2.1 and 2.2 that $0 < \sigma_r(d) < \infty$; this completes the proof. \blacksquare

3. GENERAL PERTURBATIONS

In this section we consider more general perturbations and analyze the effect of tails on deletion tolerance. In particular, we exhibit a transition occurring at a power law decay of exponent $-2d$.

Theorem 3.1. *Let Π be the perturbed lattice with perturbations Y_x with density $g(y)$.*

- If $\alpha < 2d$ and

$$\inf_{x \in \mathbb{R}^d} \frac{g(x)}{1 \wedge |x|^{-\alpha}} > 0, \quad (3.1)$$

then the perturbed lattice Π is k -deletion and k -insertion tolerant for all k and mutually absolutely continuous with Π_S for any finite set $S \subset \mathbb{Z}^d$.

- If $\alpha > 2d$ and

$$\sup_{x \in \mathbb{R}^d} \frac{g(x)}{1 \wedge |x|^{-\alpha}} < \infty, \quad (3.2)$$

then there exists ε such that the perturbed lattice with perturbations $\varepsilon' Y_x$ is k -deletion singular for all $0 < \varepsilon' < \varepsilon$ and all k . This result also holds under the condition that $\mathbb{E}|Y_x|^{\alpha-d} < \infty$.

Proof. We first establish the second half of the theorem when the tails are light. The assumption on the density in equation (3.2), or the assumption $\mathbb{E}|Y_x|^{\alpha-d} < \infty$, both imply that equation (2.5) holds for $|Y_x|_1$, so for small enough $\varepsilon > 0$, the limiting constant from (??) satisfies $M_d(|\varepsilon' Y_x|_1) < \frac{1}{2}$ for $\varepsilon' < \varepsilon$. Applying Lemma 2.2 then establishes that the perturbed lattice with perturbations $\varepsilon' Y_x$ is k -deletion singular completing the proof. The remainder of the section is devoted to establishing the first half of Theorem 3.1.

We will prove the claim in the case $k = 1$, with the extension to larger k following similarly. Let $B_r(0)$ be the Euclidian ball of radius r around the origin. Define a partition of \mathbb{R}^d into subsets $\{H_i\}_{i \geq 1}$ by $H_1 = B_2(0)$, and $H_i = B_{2^i}(0) \setminus H_{i-1}$ for $i \geq 2$.

Given that equation (3.1) holds, the density g is everywhere positive. Since the perturbations are independent, it is sufficient to show that Π_0 can be coupled with positive probability to $\hat{\Pi}$, the perturbed point process identical to Π except that the perturbation of 0 is taken according to the uniform distribution on $H_1 \cup H_2$ instead of as Y_0 . We denote these perturbations as \hat{Y}_x and will construct a coupling so that $\mathbb{P}(\hat{\Pi} = \Pi_0) > 0$.

By construction there exist constants $0 < c_1 < c_2$ such that

$$c_1 2^{id} \leq |H_i \cap (\mathbb{Z}^d \setminus \{0\})| \leq c_2 2^{id}, \quad c_1 2^{id} \leq |H_i| \leq c_2 2^{id}. \quad (3.3)$$

By equation (3.1) then we have that for some $c_3 > 0$ and for all $i \geq 1$,

$$\inf_{x \in H_i, y \in H_i \cup H_{i+1}} g(y - x) \geq c_3 2^{-\alpha i}.$$

It follows that with $c_4 = c_1 c_3$ that for all i and $x \in H_i \cup \mathbb{Z}^d \setminus \{0\}$ we can decompose the measure of $x + Y_x$ into a mixture of the uniform distribution on $H_i \cup H_{i+1}$ with probability $p_i = c_4 2^{i(d-\alpha)}$ and another probability measure μ_x with probability $1 - p_i$.

The first step of our coupling is to construct independent Bernoulli random variables $\{\zeta_x\}_{x \in \mathbb{Z} \setminus \{0\}}$ where $\mathbb{P}(\zeta_x = 1) = p_i$ when $x \in H_i$. When $\zeta_x = 0$ we choose $x + \hat{Y}_x$ according to μ_x and let $x + \hat{Y}_x = x + Y_x$ so it remains to couple the vertices with $\zeta_x = 1$ which are distributed uniformly on $H_i \cup H_{i+1}$. Let $\mathcal{Z} = \{Z_i\}_i$ where Z_i denotes the number of $x \in H_i \cap (\mathbb{Z} \setminus \{0\})$ with $\zeta_x = 1$. Counting the fact that \hat{Y}_0 is uniform on $H_1 \cup H_2$ set $\hat{Z}_1 = 1 + Z_1$ and $\hat{Z}_i = Z_i$ for $i \geq 2$. In summary the remaining not yet coupled points in Π_0 (respectively $\hat{\Pi}$) are Z_i (resp. \hat{Z}_i) points independent and uniform in $H_i \cup H_{i+1}$ for $i \geq 1$.

Now sampling according to the uniform distribution on $H_i \cup H_{i+1}$ is equivalent to first selecting H_i or H_{i+1} with probability proportional to their area and then sampling the selected region uniformly. So set $r_i = \frac{|H_i|}{|H_i \cup H_{i+1}|}$ and note that the r_i are uniformly bounded away from 0 and 1. Hence we define as binomials, $W_i = B(Z_i, r_i)$ and set $\mathcal{U} = \{U_i\}_i$ so that $U_1 = W_1$ and $U_i = W_i + Z_{i-1} - W_{i-1}$ for $i \geq 2$. Define \hat{W}_i and $\hat{\mathcal{U}} = \{\hat{W}_i\}_i$ similarly. With these definitions the remaining not yet coupled points in Π_0 (respectively $\hat{\Pi}$) are U_i (resp. \hat{U}_i) points independent and uniform in H_i for each $i \geq 1$.

So our procedure for coupling the remaining points is as follows. Given \mathcal{Z} , we take the coupling maximizing the probability that $\mathcal{U} \equiv \hat{\mathcal{U}}$. Conditional on this event the remaining points in Π_0 and $\hat{\Pi}$ have the same law, namely the union of U_i independent uniformly chosen points in H_i for each $i \geq 1$. Thus on the event $\mathcal{U} \equiv \hat{\mathcal{U}}$ we can couple Π_0 and $\hat{\Pi}$ and hence to show deletion tolerance it remains to establish that we can couple \mathcal{U} and $\hat{\mathcal{U}}$ with positive probability.

Claim 3.2. *With the definitions above, $\mathbb{P}(\mathcal{U} \equiv \hat{\mathcal{U}}) > 0$.*

With $c_5 = \frac{1}{2}c_1c_4 > 0$ denote by \mathcal{E} the event that,

$$Z_i \geq 1 \vee c_5 2^{i(2d-\alpha)}, \quad i \geq 1. \quad (3.4)$$

We will show that $\mathbb{P}(\mathcal{U} \equiv \hat{\mathcal{U}} \mid \mathcal{Z}, \mathcal{E}) > 0$. First, we will check that $\mathbb{P}(\mathcal{E}) > 0$. By construction each Z_i is independent with distribution $B(H_i \cap (\mathbb{Z} \setminus \{0\}), p_i)$ and so by equation (3.3) we have that $\mathbb{E}Z_i \geq c_1c_4 2^{i(2d-\alpha)}$. Hence with our choice of c_5 by the Azuma-Hoeffding inequality,

$$\mathbb{P}(Z_i \leq c_5 2^{i(2d-\alpha)}) \leq c_6 \exp[-c_7 2^{i(2d-\alpha)}],$$

for large i . Given this (better than) exponential decay and as $\mathbb{P}(Z_i \geq 1 \vee c_5 2^{i(2d-\alpha)}) > 0$ for all i it follows that $\mathbb{P}(\mathcal{E}) > 0$.

Now for $\mathcal{U} \equiv \hat{\mathcal{U}}$ we must have $W_1 = \hat{W}_1$ and $W_i = 1 + \hat{W}_i$ for all $i \geq 2$. The optimal coupling is at least as good as taking the optimal coupling

independently for each i so we have that

$$\mathbb{P}(\mathcal{U} \equiv \hat{\mathcal{U}} \mid \mathcal{Z}) \geq (1 - d_{\text{TV}}(W_1, \hat{W}_1 \mid \mathcal{Z})) \prod_{i=2}^{\infty} (1 - d_{\text{TV}}(W_i, \hat{W}_i + 1 \mid \mathcal{Z})) \quad (3.5)$$

where $d_{\text{TV}}(\cdot, \cdot \mid \mathcal{Z})$ denotes the total variation distance given \mathcal{Z} . Since $d_{\text{TV}}(W_1, \hat{W}_1 \mid \mathcal{Z}) < 1$ and $d_{\text{TV}}(W_i, \hat{W}_i + 1 \mid \mathcal{Z}, \mathcal{E}) < 1$ for all $i \geq 2$ it is sufficient to show that

$$\sum_{i=2}^{\infty} d_{\text{TV}}(W_i, \hat{W}_i + 1 \mid \mathcal{Z}, \mathcal{E}) < \infty. \quad (3.6)$$

Hence we estimate $d_{\text{TV}}(B(n, p), B(n, p) + 1)$. When $p \leq \frac{1}{2}$ and $|j - np| \leq (np)^{3/4}$ then

$$\begin{aligned} & \frac{|\mathbb{P}(B(n, p) = j) - \mathbb{P}(B(n, p) = j - 1)|}{\mathbb{P}(B(n, p) = j)} \\ &= \frac{\left| \binom{n}{j} p^j (1-p)^{n-j} - \binom{n}{j-1} p^{j-1} (1-p)^{n-j+1} \right|}{\binom{n}{j} p^j (1-p)^{n-j}} \\ &= \left| 1 - \frac{j}{np} \cdot \frac{n(1-p)}{(n-j+1)} \right| \\ &= \left| 1 - \left(1 + \frac{j-np}{np} \right) \left(1 - \frac{j-np-1}{n(1-p)} \right)^{-1} \right| \\ &\leq c_8 (np)^{-1/4} \end{aligned} \quad (3.7)$$

provided np and $n(1-p)$ are sufficiently large. It follows that

$$\begin{aligned} d_{\text{TV}}(B(n, p), B(n, p) + 1) &= \frac{1}{2} \sum_{j=0}^{n+1} |\mathbb{P}(B(n, p) = j) - \mathbb{P}(B(n, p) = j - 1)| \\ &\leq \mathbb{P}(|B(n, p) - np| \geq (np)^{3/4}) \\ &\quad + c_8 (np)^{-1/4} \sum_{j=np-(np)^{3/4}}^{np+(np)^{3/4}} \mathbb{P}(B(n, p) = j) \\ &\leq 2 \exp\left(-\frac{(np)^{3/2}}{n}\right) + c_8 (np)^{-1/4} \end{aligned} \quad (3.8)$$

where the first inequality is by equation (3.7) and the second is by Azuma's inequality. Since r_i is uniformly bounded away from 0 and 1 then

$$\begin{aligned} d_{\text{TV}}(W_i, \hat{W}_i + 1 \mid \mathcal{Z}) &= d_{\text{TV}}(B(Z_i, r_i), B(Z_i, r_i) + 1 \mid \mathcal{Z}) \\ &\leq 2 \exp\left(-\frac{(Z_i r_i)^{3/2}}{Z_i}\right) + c_8 (Z_i r_i)^{-1/4} \end{aligned} \quad (3.9)$$

Substituting (3.9) into equation (3.6) we have that

$$\begin{aligned} & \sum_{i=2}^{\infty} d_{\text{TV}}(W_i, \hat{W}_i + 1 \mid \mathcal{Z}, \mathcal{E}) \\ & \leq \sum_{i=2}^{\infty} 2 \exp \left(-(1 \vee c_5 2^{i(2d-\alpha)})^{1/2} r_i^{3/2} \right) + c_8 ((1 \vee c_5 2^{i(2d-\alpha)}) r_i)^{-1/4} < \infty, \end{aligned}$$

which establishes that $\mathbb{P}(\mathcal{U} \equiv \hat{\mathcal{U}} \mid \mathcal{Z}, \mathcal{E}) > 0$ completing the claim.

Thus the claim ensures we can couple \mathcal{U} and $\hat{\mathcal{U}}$ with positive probability which completes the coupling of Π_0 and $\hat{\Pi}$ and proves that Π and Π_0 are not mutually singular. Then the deletion and insertion tolerance of Π and its mutual absolute continuity with Π_0 follow by Proposition 1.5. ■

4. PROOF OF THEOREM 1.4

The proof of Theorem 1.4 is essentially complete from previous results. When $\alpha > 1$ then the perturbations have finite first moments and the deletion intolerance result of [10]. When $\alpha < 1$ the perturbations satisfy (3.1) and so deletion and insertion tolerance follow from Theorem 3.1. The sole remaining case is to show that Π is deletion singular when $\alpha = 1$ (Cauchy perturbations) which is verified as a special case in the following subsection.

4.1. Cauchy perturbations.

Lemma 4.1. *If $d = 1$ and $\{Y_x\}$ are i.i.d. Cauchy distributed, then the perturbed lattice is k -deletion singular for all k .*

Proof. Our proof follows the approach of [10]. Let $S \subset \mathbb{Z}^d$ and $\Phi_{m,x} = \max\{m - |x + Y_x|, 0\}$. We define

$$\Psi_m(\Pi) = \frac{1}{m} \int_{-m}^m (m - |z|) \Pi(dz) = \frac{1}{m} \sum_{x \in \mathbb{Z}} \Phi_{m,x}.$$

We similarly have

$$\Psi_m(\Pi_S) = \frac{1}{m} \int_{-m}^m (m - |z|) \Pi_S(dz) = \frac{1}{m} \sum_{x \in \mathbb{Z} \setminus S} \Phi_{m,x}.$$

and so

$$\Psi_m(\Pi) - \Psi_m(\Pi_0) = \frac{1}{m} \sum_{x \in S} \Phi_{m,x} \rightarrow |S| \quad (4.1)$$

almost surely. We next consider the variance of $\Psi_m(\Pi)$ which is bounded as

$$\text{Var}(\Phi_{m,x}) \leq \mathbb{E}[\Phi_{m,x} - \max\{m - |x|, 0\}]^2 \leq \mathbb{E}[|Y_x| \wedge m]^2 \leq Cm. \quad (4.2)$$

If $|x| > 2m$ then since $|\Phi_{m,x}| \leq m$ and since the density of the Cauchy decays like cy^{-2} ,

$$\begin{aligned} \text{Var}(\Phi_{m,x}) &\leq \mathbb{E}[\Phi_{m,x}]^2 \leq m^2 \mathbb{P}(Y_x \in [-m-x, m-x]) \\ &\leq Cm^2 \int_{-m-x}^{m-x} y^{-2} dy \leq C'm^3|x|^{-2}. \end{aligned} \quad (4.3)$$

Since the $\Phi_{m,x}$ are independent (over x) combining equations (4.2) and (4.3) we have that

$$\text{Var} \Psi_m(\Pi) \leq \frac{1}{m^2} \left[\sum_{x=-2m}^{2m} Cm + \sum_{|x|>2m} C'm^3|x|^{-2} \right] = O(1). \quad (4.4)$$

Now for $m' > m$ we calculate the covariance of $\Psi_m(\Pi)$ and $\Psi_{m'}(\Pi)$ as

$$\begin{aligned} \text{Cov}(\Psi_m, \Psi_{m'}(\Pi)) &= \frac{1}{mm'} \sum_x \text{Cov}(\Phi_{m,x}, \Phi_{m',x}) \\ &\leq \frac{1}{mm'} \sum_x \sqrt{\text{Var}(\Phi_{m,x}) \text{Var}(\Phi_{m',x})} \\ &\leq \frac{1}{mm'} \left[\sum_{x=-2m}^{2m} \sqrt{C^2mm'} + \sum_{2m<|x|\leq 2m'} \sqrt{CC'm^3m'|x|^{-2}} \right. \\ &\quad \left. + \sum_{|x|>2m'} C'm^{3/2}m'^{3/2}|x|^{-2} \right] \\ &\leq (m/m')^{1/2} [C_1 + C_2 \log(m'/m) + C_3] \\ &\leq C_4(m/m')^{1/2} \log(m'/m). \end{aligned}$$

Then if we take $m_\ell = e^{2\ell^2}$ we have that $\text{Cov}(\Psi_{m_\ell}, \Psi_{m_{\ell'}}(\Pi)) \leq O(e^{-(\ell \vee \ell')})$ when $\ell \neq \ell'$ and hence

$$\text{Var} \left[\frac{1}{n} \sum_{\ell=1}^n \Psi_{m_\ell}(\Pi) \right] = o(1).$$

So we have that $\frac{1}{n} \sum_{i=1}^n \Psi_{m_i}(\Pi) - \mathbb{E} \Psi_{m_i}(\Pi)$ converges to 0 in probability while by (4.1) we have that $\frac{1}{n} \sum_{i=1}^n \Psi_{m_i}(\Pi_0) - \mathbb{E} \Psi_{m_i}(\Pi)$ converges to $-|S|$ in probability. It follows that Π and Π_S are mutually singular and so by Proposition 6.1 Π is k -deletion singular for all k . \blacksquare

5. ABSOLUTE CONTINUITY

In this section we prove the equivalences of deletion intolerance and insertion intolerance and deletion and insertion singularity.

Proof of Proposition 1.5. In this section we establish Proposition 1.5. Let \mathbb{Q} denote the law of Π and for a finite set $S \subset \mathbb{Z}^d$ we denote $\Pi_S = \{x + Y_x : x \in \mathbb{Z}^d \setminus S\}$ and its law as \mathbb{Q}_S .

(5) \iff (6). If Π and Π_0 are mutually singular then clearly Π and Π_0 are not mutually absolutely continuous. Now assume that Π and Π_0 are not mutually singular but that \mathbb{Q}_0 is not absolutely continuous with respect to \mathbb{Q} . Then we can find a measurable set A such that $\mathbb{Q}(A) = 1$ and $0 < \mathbb{Q}_0(A) < 1$ and that on A , \mathbb{Q}_0 is absolutely continuous with respect to \mathbb{Q} with a Radon-Nikodym derivative given by $\kappa(a) = \frac{d\mathbb{Q}_0}{d\mathbb{Q}}$. We will show that $\Pi_0 \in A$ is a tail event for the $\{Y_x\}$.

For some $S \subset \mathbb{Z}^d \setminus \{0\}$ define the set

$$B = B_S := \{b : \mathbb{Q}_0(A \mid \Pi_{S \cup \{0\}} = b) \in (0, 1)\}.$$

Suppose that $\mathbb{P}[\Pi_{S \cup \{0\}} \in B] > 0$. Defining the sub-probability measure

$$\tilde{\mathbb{Q}}_0(E) := \mathbb{P}[\Pi_0 \in E \cap A, \Pi_{S \cup \{0\}} \in B],$$

we have that

$$\tilde{\mathbb{Q}}_0(A) := \mathbb{P}[\Pi_0 \in A, \Pi_{S \cup \{0\}} \in B] = \int_B \mathbb{Q}_0(A \mid \Pi_{S \cup \{0\}} = b) d\mathbb{Q}_{S \cup \{0\}}(b) > 0.$$

Since $\tilde{\mathbb{Q}}_0$ is dominated by \mathbb{Q}_0 it is absolutely continuous with respect to \mathbb{Q} and so not mutually singular. Hence there exists a coupling of (Y_x, Π) and an identically distributed copy (Y_x^*, Π^*) such that

$$\mathbb{P}[\Pi = \Pi^*, \Pi_{S \cup \{0\}}^* \in B] > 0.$$

Since the points $\{x + Y_x^* : x \in S\}$ must have images in Π when the point processes are equal and as there are only countably many choices of $\hat{S} \subset \mathbb{Z}^d$ with $|\hat{S}| = |S|$ we have that for some \hat{S} ,

$$\mathbb{P}[\Pi_{\hat{S}} = \Pi_{S \cup \{0\}}^*, \Pi_{S \cup \{0\}}^* \in B] > 0.$$

As the Y_x have a positive density everywhere the sets $\{x + Y_x^* : x \in S\}$ and $\{x + Y_x : x \in \hat{S}\}$ are mutually absolutely continuous and hence the distributions $\mathbb{Q}_0(\cdot \mid \Pi_{S \cup \{0\}} = b)$ and $\mathbb{Q}(\cdot \mid \Pi_{\hat{S}} = b)$ are also mutually absolutely continuous. Then by definition of B we have that for all $b \in B$, $\mathbb{Q}(A \mid \Pi_{\hat{S}} = b) < 1$ and hence

$$\mathbb{Q}(A^c) \geq \mathbb{P}[\Pi_{\hat{S}} \in B, \Pi \notin A] = \int_B \mathbb{Q}(A^c \mid \Pi_{S \cup \{0\}} = b) d\mathbb{Q}_{\hat{S}}(b) > 0.$$

But $\mathbb{Q}(A^c) = 0$ so we have a contradiction and hence $\mathbb{P}[\mathbb{P}(\Pi_0 \in A \mid \Pi_{S \cup \{0\}}) \in (0, 1)] = 0$ for all S . This implies that $\Pi_0 \in A$ is a tail event and so by the Kolmogorov zero-one law we have that $\mathbb{P}[\Pi_0 \in A] = 1$ since $\mathbb{Q}_0(A) > 0$. This contradicts our assumption that $\mathbb{Q}_0(A) < 1$ so we have that \mathbb{Q}_0 is absolutely

continuous with respect to \mathbb{Q} . That \mathbb{Q} is absolutely continuous with respect to \mathbb{Q}_0 follows similarly so the laws are mutually absolutely continuous.

(1) \iff (3) \iff (6). Suppose \mathbb{Q} and \mathbb{Q}_0 are singular. If Z is a Π point then by an abuse of notation let Π_Z denote $\Pi \setminus Z$. Let $X \in \mathbb{Z}^d$ be the random lattice point such that $X + Y_X = Z$ so $\Pi_Z = \Pi_X$. Since, by translation, each Π_x is singular to Π so is Π_X because $X \in \mathbb{Z}^d$ which is countable. Hence Π_Z is mutually singular to Π and so Π is deletion singular and hence also deletion intolerant.

Conversely, suppose that \mathbb{Q} and \mathbb{Q}_0 are not mutually singular, so they must be mutually absolutely continuous. Then for any set A , if $\mathbb{P}[\Pi \in A] = 0$ then $\mathbb{P}[\Pi_x \in A] = 0$, whence $\mathbb{P}[\Pi_Z \in A] \leq \sum_{x \in \mathbb{Z}^d} \mathbb{P}[\Pi_x \in A] = 0$. Thus Π_Z is absolutely continuous with respect to Π , so Π is deletion tolerant and not deletion singular.

(2) \iff (4) \iff (6). Suppose \mathbb{Q} and \mathbb{Q}_0 are mutually singular. Let $V \subset \mathbb{R}^d$ be a Borel set with Lebesgue measure $\mathcal{L}(V) \in (0, \infty)$ and U a random variable independent of $\{Y_x\}_{x \in \mathbb{Z}^d}$ uniform on V . Suppose that $\Pi \cup U$ is not mutually singular with respect to Π . Then there exists an identically distributed copy (U^*, Y_x^*, Π^*) and a coupling such that

$$\mathbb{P}[\Pi^* \cup U^* = \Pi] > 0.$$

On the event that they agree let X denote the random lattice point such that $X + Y_X = U^*$. For some x we have $\mathbb{P}[X = x, \Pi^* \cup U^* = \Pi] > 0$ and hence $\mathbb{P}[\Pi^* = \Pi_x] > 0$. But \mathbb{Q} and \mathbb{Q}_x are mutually singular which is a contradiction so $\Pi \cup V$ is mutually singular with respect to Π and hence Π is insertion singular and hence insertion intolerant.

Conversely if \mathbb{Q} and \mathbb{Q}_0 are mutually absolutely continuous then (U, Π) is absolutely continuous with respect to (Y_0, Π_0) since Y_0 has a positive density everywhere. It follows that $\Pi \cup U$ is absolutely continuous with respect to $Y_0 \cup \Pi_0 = \Pi$ so Π is insertion tolerant and hence not insertion singular. ■

6. RIGIDITY

We begin with the following Proposition relating the k -deletion tolerance versions and which follows with minor modification to the proof of Proposition 1.5.

Proposition 6.1. *If the distribution of Y_x has a density which is everywhere positive and $S \subset \mathbb{Z}^d$ is of size k then the following are equivalent*

- (1) *The perturbed lattice is k -deletion tolerant.*
- (2) *The perturbed lattice is k -insertion tolerant.*
- (3) *The perturbed lattice is not k -deletion singular.*

- (4) *The perturbed lattice is not k -insertion singular.*
- (5) *The measures Π and Π_S are mutually absolutely continuous.*
- (6) *The measures Π and Π_S are not mutually singular.*

The proof of Proposition 6.1 follows by the same proof as Proposition 1.5 with the minor alteration of adding or removing k -points instead of 1. Finally we prove Proposition 1.2 relating rigidity and deletion tolerance.

Proof. (Proof of Proposition 1.2) Suppose first that there exists some S such that Π_S is not singular with respect to Π but that Π is rigid. Then $N(\Pi_{\text{out}}) = |\Pi_{\text{in}}|$ a.s. but also $N((\Pi_S)_{\text{out}}) = |(\Pi_S)_{\text{in}}|$ a.s. since Π is mutually absolutely continuous with respect to Π_S by Proposition 6.1. However, on the event $A = \{\forall x \in S : x + Y_x \in B_1(0)\}$ by definition $\Pi_{\text{out}} = (\Pi_S)_{\text{out}}$ but $\Pi_{\text{in}} = (\Pi_S)_{\text{in}} + |S|$. Since $\mathbb{P}[A] > 0$ this is a contradiction and so Π is not rigid.

Now suppose that Π is not rigid and fix some ball B for which it fails. Let $\psi(\Pi_{\text{out}}, j) = \mathbb{P}[|\Pi_{\text{in}}| = j \mid \Pi_{\text{out}}]$. Since Π is not rigid it follows that

$$\mathbb{P}[\psi(\Pi_{\text{out}}, \Pi_{\text{in}}) < 1] > 0$$

and since $\mathbb{E}[|\Pi_{\text{in}}| \mid \Pi_{\text{out}}] = \sum_j j \psi(\Pi_{\text{out}}, j)$ we have that

$$\mathbb{P}\left[\sum_{j < \Pi_{\text{in}}} \psi(\Pi_{\text{out}}, j) > 0\right] > 0.$$

In particular for some positive integer k we have that

$$\mathbb{P}[\psi(\Pi_{\text{out}}, \Pi_{\text{in}} - k) > 0] > 0.$$

Thus we can construct an independent copy Π' of Π such that

$$\mathbb{P}[\Pi'_{\text{out}} = \Pi_{\text{out}}, |\Pi'_{\text{in}}| + k = |\Pi_{\text{in}}|] > 0.$$

Now since there are a countable number of finite subsets of \mathbb{Z}^d we can find sets $S, S' \subset \mathbb{Z}^d$ with $|S| = |S'| + k$

$$\mathbb{P}[\Pi'_{\text{out}} = \Pi_{\text{out}}, \{x + Y'_x : x \in S'\} = \Pi'_{\text{in}}, \{x + Y_x : x \in S\} = \Pi_{\text{in}}] > 0$$

and so by removing these points

$$\mathbb{P}[\Pi'_{S'} = \Pi_S] > 0. \tag{6.1}$$

Let $S^* \subset S$ with $|S^*| = k$ and let $U_1, \dots, U_{|S'|}$ be i.i.d. standard d -dimensional Gaussians. Then since each U_i is mutually absolutely continuous with respect to $x + Y_x$ for any x then $\Pi_S \cup \{U_1, \dots, U_{|S'|}\}$ is mutually absolutely continuous with respect to Π_{S^*} and $\Pi'_S \cup \{U_1, \dots, U_{|S'|}\}$ is mutually absolutely continuous with respect to Π' and hence Π . Combining this with (6.1) implies that Π and Π_{S^*} are not mutually singular which completes the proof. \blacksquare

7. DELETION SINGULARITY WITHOUT RIGIDITY

In this section we prove Theorem 1.6.

Proof. First we show that $\hat{\Pi}$ is 2-deletion tolerant if $\sigma^2 - \delta^2 > \sigma_c^2$. By Theorem 1.3 we have that $\Pi = \{x + Y_x : x \in \mathbb{Z}^d\}$ is deletion tolerant. We can construct $\hat{\Pi}$ from Π by replacing each point in $z \in \Pi$ with points $z + G_{z,1}$ and $z + G_{z,2}$ for independent $N_d(0, \delta^2)$ Gaussians $G_{z,1}$ and $G_{z,2}$. Since Π and Π_0 are mutually absolutely continuous, by Proposition 1.5 we have that $\hat{\Pi}$ and $\hat{\Pi}_0 = \{x + \hat{Y}_{x,i} : (x, i) \in (\mathbb{Z}^d \setminus \{0\}) \times \{1, 2\}\}$ are mutually absolutely continuous. Arguing similarly to the proof of Proposition 1.5 it follows $\hat{\Pi}$ is 2-deletion tolerant.

To prove that $\hat{\Pi}$ is not deletion tolerant we again argue by contradiction from a coupling as in the proof of Lemma 2.2.

Let \mathcal{V}_n be the set of all pairs of sequences $V_n = ((v_0, \dots, v_n), (v_0^*, \dots, v_n^*))$ taking elements in $\mathbb{Z}^d \times \{1, 2\}$ such that the elements $v_0(1), \dots, v_n(1)$ are distinct as are $v_0^*(1), \dots, v_n^*(1)$ where $v_0^* = (0, 2)$ and with $v_0^* = (0, 2)$. Let

$$L_n = L_n(V_n) = \sum_{i=0}^n |v_i^*(1) - v_i(1)|_1 + \sum_{i=0}^{n-1} |v_i(1) - v_{i+1}^*(1)|_1.$$

Note that since the $v_i(0)$ are distinct we have that $L_n \geq n$. We define the following collection of events for constants $C_1(d) > 0$ to be fixed later

- Let $\mathcal{I}_n(V_n)$ (respectively $\mathcal{I}_n^*(V_n)$) be the event that

$$\sum_{i=0}^n \sum_{j=1}^2 |\hat{Y}_{v_i(1),j}|_1 \geq \frac{1}{2} L_n, \quad \text{resp.} \quad \sum_{i=0}^n \sum_{j=1}^2 |\hat{Y}_{v_i^*(1),j}|_1 \geq \frac{1}{2} L_n.$$

- Let $\mathcal{J}_n(V_n)$ be the event that

$$\sum_{i=0}^{n-1} I(|\hat{Y}_{v_i(1),1} - \hat{Y}_{v_i(1),2}|_1 \geq C_1) \geq \frac{1}{2} n,$$

where $I(\cdot)$ denotes the indicator.

- Let $\mathcal{J}_n^*(V_n)$ be the event that

$$\sum_{i=0}^{n-1} I(|(v_i^*(1) + \hat{Y}_{v_i^*(1),3-v_i^*(2)}) - (v_{i+1}^*(1) + \hat{Y}_{v_{i+1}^*(1),v_{i+1}^*(2)})|_1 \leq C_1) \geq \frac{1}{2} n.$$

By basic large deviations estimates since $\hat{Y}_{v_i(1),j}$ are $N_d(0, \sigma^2)$ then for sufficiently large $C_2(d)$ when $L_n \geq C_2 n$,

$$\mathbb{P}[\mathcal{I}_n(V_n)] \leq (4d)^{-L_n}. \quad (7.1)$$

and similarly for $\mathcal{I}_n^*(V_n)$. Let \mathcal{F}_i be the σ -algebra generated by $\{\hat{Y}_{v_i^*(1),1}^*, \hat{Y}_{v_i^*(1),1}^*\}_{1 \leq i' \leq i}$. By choosing $C_1 = C_1(d, \sigma)$ to be sufficiently small we can make

$$\mathbb{P}[|(v_i^*(1) + \hat{Y}_{v_i^*(1),3-v_i^*(2)}^*) - (v_{i+1}^*(1) + \hat{Y}_{v_{i+1}^*(1),v_{i+1}^*(2)}^*)|_1 \leq C_1 \mid \mathcal{F}_i] < \frac{1}{4}(4d)^{-2C_2},$$

for all V_n and i since $\hat{Y}_{v_{i+1}^*(1),v_{i+1}^*(2)}^*$ is distributed as $N_d(0, \sigma^2)$ and is independent of \mathcal{F}_i . Hence

$$\mathbb{P}[\mathcal{J}_n^*(V_n)] \leq \binom{n}{n/2} \left(\frac{1}{4}(4d)^{-2C_2} \right)^{\frac{1}{2}n} \leq (4d)^{-C_2 n}, \quad (7.2)$$

for large enough n . Finally, we may choose $\delta > 0$ to be sufficiently small so that

$$\mathbb{P}[|\hat{Y}_{v_i(1),1} - \hat{Y}_{v_i(1),2}|_1 \geq C_1] \leq \frac{1}{4}(4d)^{-C_2},$$

since $\hat{Y}_{v_i(1),1} - \hat{Y}_{v_i(1),2}$ is distributed as $N_d(2\delta^2)$ and hence

$$\mathbb{P}[\mathcal{J}_n(V_n)] \leq (4d)^{-C_2 n}. \quad (7.3)$$

Finally we note that $\{V_n \in \mathcal{V}_n : L_n(V_n) = \ell\} \leq (2d)^{L_n}$.

Now suppose that $\hat{\Pi}$ and $\hat{\Pi}_{(0,1)} = \{x + \hat{Y}_{x,i} : (x,i) \in (\mathbb{Z}^d \times \{1,2\}) \setminus \{(0,1)\}\}$ are not mutually singular. Then there exists an identically distributed copy $(\{\hat{Y}_{x,i}^*\}, \Pi^*)$ and a coupling so that the event $\mathcal{A} = \{\Pi = \hat{\Pi}_{(0,1)}^*\}$ has positive probability. We define the bijections $W : \hat{\Pi} \rightarrow \mathbb{Z}^d \times \{1,2\}$ and $W^* : \hat{\Pi}_{(0,1)}^* \rightarrow \mathbb{Z}^d \times \{1,2\} \setminus \{(0,1)\}$ so that

$$y = W_1(y) + \hat{Y}_{W(y)}, \quad y = W_1^*(y) + \hat{Y}_{W^*(y)}.$$

On \mathcal{A} define the sequence $u_0^* = (0,1)$ and

$$u_i = W(u_i^*(1) + \hat{Y}_{u_i^*(1),3-u_i^*(2)}^*), \quad u_{i+1}^* = W^*(u_i(1) + \hat{Y}_{u_i(1),3-u_i(2)}),$$

where $u_i(1)$ denotes the first coordinate of u_i . By construction the $\{u_i(1)\}_{i \geq 0}$ are distinct as are the $\{u_i^*(1)\}_{i \geq 0}$ and $U_n = ((u_0, \dots, u_n), (u_0^*, \dots, u_n^*)) \in \mathcal{V}_n$. Also by construction

$$\begin{aligned} u_i^*(1) + \hat{Y}_{u_i^*(1),3-u_i^*(2)}^* &= u_i(1) + \hat{Y}_{u_i(1),u_i(2)}, \\ u_i(1) + \hat{Y}_{u_i(1),3-u_i(2)} &= u_{i+1}^*(1) + \hat{Y}_{u_{i+1}^*(1),u_{i+1}^*(2)}, \end{aligned}$$

and hence by the triangle inequality we have that

$$\begin{aligned} L_n(V_n) &= \sum_{i=0}^n |u_i^*(1) - u_i(1)|_1 + \sum_{i=0}^{n-1} |u_i(1) - u_{i+1}^*(1)|_1 \\ &\leq \sum_{i=0}^n \sum_{j=1}^2 |\hat{Y}_{u_i(1),j}|_1 + \sum_{i=0}^n \sum_{j=1}^2 |\hat{Y}_{u_i^*(1),j}|_1 \end{aligned}$$

and so the event $\mathcal{I}_n(U_n) \cup \mathcal{I}_n^*(U_n)$ holds on \mathcal{A} .

Again by the definition of U_n ,

$$\hat{Y}_{v_i(1),1} - \hat{Y}_{v_i(1),2} = (v_i^*(1) + \hat{Y}_{v_i^*(1),3-v_i^*(2)}^*) - (v_{i+1}^*(1) + \hat{Y}_{v_{i+1}^*(1),v_{i+1}^*(2)}^*)$$

and hence at least on of $\mathcal{J}_n(U_n)$ and $\mathcal{J}_n^*(U_n)$ holds on \mathcal{A} . Hence

$$\begin{aligned} \mathbb{P}[\mathcal{A}] &\leq \sum_{V_n \in \mathcal{V}_n} \mathbb{P}[(\mathcal{I}_n(V_n) \cup \mathcal{I}_n^*(V_n)) \cap (\mathcal{J}_n(V_n) \cup \mathcal{J}_n^*(V_n))] \\ &\leq \sum_{m=n}^{C_2 n} \sum_{\substack{V_n \in \mathcal{V}_n \\ L_N(V_n)=m}} \mathbb{P}[\mathcal{J}_n(V_n) \cup \mathcal{J}_n^*(V_n)] \\ &\quad + \sum_{m=C_2 n}^{\infty} \sum_{\substack{V_n \in \mathcal{V}_n \\ L_N(V_n)=m}} \mathbb{P}[\mathcal{I}_n(V_n) \cup \mathcal{I}_n^*(V_n)] \end{aligned}$$

By equation (7.1)

$$\sum_{m=C_2 n}^{\infty} \sum_{\substack{V_n \in \mathcal{V}_n \\ L_N(V_n)=m}} \mathbb{P}[\mathcal{I}_n(V_n) \cup \mathcal{I}_n^*(V_n)] \leq \sum_{m=C_2 n}^{\infty} (2d)^m (4d)^{-m},$$

and by equation (7.3) and (7.2)

$$\sum_{m=n}^{C_2 n} \sum_{\substack{V_n \in \mathcal{V}_n \\ L_N(V_n)=m}} \mathbb{P}[\mathcal{J}_n(V_n) \cup \mathcal{J}_n^*(V_n)] \leq \sum_{m=n}^{C_2 n} (2d)^m (4d)^{-C_2 n}$$

and since both of these bounds tends to 0 as n tends to infinity we have that $\mathbb{P}[\mathcal{A}] = 0$. This is a contradiction and hence $\hat{\Pi}$ is deletion singular. \blacksquare

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